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1968 J. Phys. A: Gen. Phys. 1 277

(http://iopscience.iop.org/0022-3689/1/2/314)

J. PHYS. A (PROC. PHYS. SOC.), 1968, SER. 2, VOL. 1. PRINTED IN GREAT BRITAIN

The generalized Bel–Robinson tensor as a generator

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MS. received 19*th December* 1967

Abstract. It is shown, on the basis of an explicit calculation using canonical field theory, that the generalized Bel–Robinson tensor for arbitrary spin generates a transformation which is equivalent to a multiple derivative of a translation in agreement for spin 2 with a recent calculation of Komar's.

The transformations generated by the Zilch are also briefly considered.

1. Introduction

Komar (1968) has shown in the linearized Einstein theory that the Bel–Robinson superenergy tensor $T_{\mu\nu\alpha\beta}$ (e.g. Trautman 1962) generates the canonical transformation which is obtained by taking the second derivative of a translation. He shows, in fact, that

$$\left[h_{\rho\sigma}, \int_\sigma T_{\alpha\beta\mu}{}^{\nu}\, d\sigma_\nu\right] \sim \partial_\mu \partial_\alpha \partial_\beta h_{\rho\sigma} \tag{1}$$

where $h_{\rho\sigma}$ is the small metric deviation from Cartesian values and σ is a space-like surface. Komar's calculation is a classical one and the left-hand side of (1) is to be taken as a Poisson bracket.

In this brief paper we should like to show how this result follows rather neatly using an expression for a generalization of the Bel–Robinson tensor given in earlier papers (Dowker and Dowker 1966 b, Dowker and Goldstone 1967, Dowker 1967). We shall employ the formalism of quantum field theory and will therefore derive (anti-) commutators rather than Poisson brackets.

2. Calculation

The system we consider is described by a massless $(2j+1)$-spinor φ_j. (For descriptions of the notions and notations we use, refer to our previous references and also to Dowker and Dowker 1966 a, Williams 1965, Weinberg 1964 a, b, Tung 1967.) φ_j satisfies free-field equations which can be written in the form (Dowker 1967, Dowker and Goldstone 1967)

$$t^{\mu_1 \cdots \mu_{2j}} \partial_{\mu_1} \varphi_j = 0 \tag{2}$$

where the matrices $t^{\mu_1 \cdots \mu_{2j}} \equiv t^{(\mu)}$ have been adequately described in the cited references. The generalized Bel–Robinson tensor $T_j{}^{(\mu)}$ is defined by

$$T_j{}^{(\mu)} = \varphi_j{}^{\dagger} t^{(\mu)} \varphi_j \tag{3}$$

and is conserved in view of (2).

We are now going to calculate the (anti-) commutator

$$[\varphi_j(x), P_j{}^{(\alpha)}]_{\pm} \tag{4}$$

where

$$P_j{}^{(\alpha)} = \int \varphi_j{}^{\dagger}(y) t^{(\alpha)0} \varphi_j(y)\, d^3y.$$

This is the quantum analogue of the left-hand side of (1). We have chosen the space-like surface to be $y^0 = \text{const}$. In particular, since $T_j{}^{(\mu)}$ is conserved we may take $y^0 = x^0$.

We then need the equal-time commutation rules for the φ_j. These are (Weinberg 1964 a, b, Tung 1967, Lomont 1958)

$$[\varphi_j(x), \varphi_j(y)]_{\pm} = 0 \tag{5}$$

$$[\varphi_j(x), \varphi_j{}^{\dagger}(y)]_{\pm} = \frac{1}{2p_0}\{\bar{t}^{(\mu)}p_{(\mu)} - (-1)^{2j}\bar{t}^{(\mu)}p_{(\mu)}{}'\}\delta(\mathbf{x}-\mathbf{y}) \tag{6}$$

where $p_\mu = i\partial_\mu$, $p_\mu p^\mu = 0$ and $p_\mu{}' = p^\mu$. Also $p_{(\mu)} = p_{\mu_1} \dots p_{\mu_{2j}}$.

If now (5) and (6) are substituted into (4) and the integration performed we find

$$\frac{1}{2p_0}\{\bar{t}^{(\mu)}p_{(\mu)} - (-1)^{2j}\bar{t}^{(\mu)}p_{(\mu)}{}'\}t^{(\alpha)0}\varphi_j(x). \tag{7}$$

We can further replace $\bar{t}^{(\mu)}p_{(\mu)}{}'$ by $t^{(\mu)}p_{(\mu)}$ since $\bar{t}^{(\mu)}$ differs from $t^{(\mu)}$ only in that corresponding terms with odd numbers of spatial indices have opposite signs (e.g. Weinberg 1964 a).

To prove the required theorem it is necessary to use some algebraic relations satisfied by the $t^{(\mu)}$ and $\bar{t}^{(\mu)}$. Let us consider firstly the term

$$\frac{1}{p_0}\bar{t}^{(\mu)}t^{(\alpha)0}p_{(\mu)}\varphi_j(x).$$

To reduce this, we employ the anti-commutation rule (Dowker and Dowker 1966 a, Williams 1965, Weinberg 1964 a)

$$\bar{t}^{(\mu_1\dots\mu_{2j}}t^{\nu_1\dots\nu_{2j})} = g^{(\mu_1\nu_1}g^{\mu_2\nu_2}\dots g^{\mu_{2j}\nu_{2j})} \tag{8}$$

where the parentheses signify complete symmetrization of the indicated indices. Equation (8) yields easily, if (2) is used,

$$\frac{1}{p_0}\bar{t}^{(\mu)}t^{(\alpha)0}p_{(\mu)}\varphi_j = p^{(\alpha)}\varphi_j. \tag{9}$$

There are $2j-1$ derivatives on the right-hand side of (9).

We now show that the term $t^{(\mu)}t^{(\alpha)0}p_{(\mu)}\varphi_j$ vanishes. To do this, formula (9) of Dowker and Dowker (1966 a) is convenient. This formula reads

$$t^{(\mu)}J_i = \sum_{k=1}^{2j} g^{\mu_k[m}t^{l]\mu_1\dots\mu_{k-1}\mu_{k+1}\dots\mu_{2j}}(\delta^{m0}\delta^{li} - \tfrac{1}{2}i\epsilon_{ilm}). \tag{10}$$

We now note that the matrices $t^{(\mu)}$ are polynomials of maximum degree $2j$ in the angular momentum matrices J (Weinberg 1964 a). In particular, if one index is temporal, e.g. $t^{(\alpha)0}$, a maximum product of $2j-1$ J matrices occur in this polynomial. Thus we see that repeated application of (10) leads to a final expression for the quantity $t^{(\mu)}t^{(\alpha)0}$, on each t of which there is at least one μ index, so that $t^{(\mu)}t^{(\alpha)0}p_{(\mu)}\varphi_j$ vanishes because of (2).

Finally, then, we obtain the formula

$$[\varphi_j(x), P_j{}^{(\alpha)}]_{\pm} = \tfrac{1}{2}i^{2j-1}\partial^{(\alpha)}\varphi_j(x) \tag{11}$$

which generalizes Komar's result. In the special case of gravitation *in vacuo* $j = 2$ and φ is just a complex, 5-spinor arrangement of the ten, real independent components of the Weyl conformal tensor. Equation (2) is then equivalent to the Bianchi identity.

3. Conclusion and discussion

Equation (11) is the essential result of the present work. However, we can go on and enquire whether, by the same method of using canonical field theory, we can obtain the transformations generated by the conserved Zilch-like tensors. As was shown by Dowker (1967), the simplest and original Zilch (Lipkin 1964) is generalized for arbitrary spin to the quantity

$$Z^{(\alpha)\beta} = \tfrac{1}{2} i\varphi_j{}^{\dagger} t^{(\alpha)} \overleftrightarrow{\partial^{\beta}} \varphi_j.$$

This object is traceless and is conserved on all indices. It is thus natural to ask about the transformations generated by the constants Z_1 and Z_2, where

$$Z_1{}^{(\alpha)} \equiv \int Z_1{}^{(\alpha)0}(y)\, d^3y, \qquad Z_2{}^{(\alpha-1)\beta} \equiv \int Z^{(\alpha-1)0\beta}(y)\, d^3y.$$

Using the equal time commutation rules

$$[\varphi_j(x), \dot{\varphi}_j{}^{\dagger}(y)]_{\pm} = \tfrac{1}{2} i\{\bar{t}^{(\mu)} p_{(\mu)} + (-1)^{2j} \bar{t}^{(\mu)} p_{(\mu)}{}'\}\delta(\mathbf{x}-\mathbf{y})$$

$$[\varphi_j(x), \partial_y{}^i \varphi_j{}^{\dagger}(y)]_{\pm} = -\frac{1}{2p_0}\{\bar{t}^{(\mu)} p_{(\mu)} - (-1)^{2j} \bar{t}^{(\mu)} p_{(\mu)}{}'\}\partial_x{}^i \delta(\mathbf{x}-\mathbf{y})$$

an almost identical calculation with the previous one yields

$$[\varphi_j(x), Z_1{}^{(\alpha)}]_{\pm} = \tfrac{1}{2} i^{2j} \partial^{(\alpha)} \varphi_j(x)$$

$$[\varphi_j(x), Z_2{}^{(\alpha-1)\beta}]_{\pm} = \tfrac{1}{2} i^{2j} \partial^{(\alpha-1)} \partial^{\beta} \varphi_j(x).$$

These results are identical with those of Steudel (1965), whose calculation is based on Noether's theorem in a classical Lagrangian approach.

The possible significance of these results for the programme of the quantization of general relativity has been discussed by Komar (1968). The basic idea is that in flat space there exists a preferred set of canonical transformations—those generated by the symmetry group of space–time; since the energy–momentum tensor generates translations in space–time, perhaps we can use the superenergy tensor in a similar way in curved space–time to introduce a preferred set of canonical transformations. Whether this hope is born out remains to be seen.

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